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ON (DE)HOMOGENIZED GRÖBNER BASES

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Abstract

In a relatively extensive context, (de)homogenized Gröbner bases are studied systematically. The obtained results reveal further applications of Gröbner bases to the structure theory of algebras.

1. Introduction

Let K be a field. In the computational Gröbner basis theory for a commutative polynomial algebra $K[x_1, \ldots, x_n]$ or for a non-commutative free algebra $K\langle X_1, \ldots, X_n\rangle$, it is a well-known fact that a homogenous Gröbner basis is easier to be obtained, that is, by virtue of both the structural advantage (mainly the degree-truncated structure) and the computational advantage (mainly the use of a degree-preserving fast ordering), most of the practically used commutative or non-commutative Gröbner basis algorithms produce a Gröbner basis by homogenizing generators first (if the given generators are not homogeneous), and then,

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in turn, producing a homogeneous Gröbner basis and dehomogenizing it. On the other hand, Gröbner bases and the (de)homogenization (i.e., homogenization and dehomogenization) techniques have been combined in ([7], [9], [10], [11]) to study some global structure properties of algebras defined by relations. Following the idea of ([7], [9], [10], [13]) concerning (de)homogenized Gröbner bases, in this paper, we systematize and deepen the study on this topic. More precisely, after giving a quick introduction to Gröbner bases for ideals in an algebra with a skew multiplicative K-basis in Section 1, we employ the (de)homogenization technique as used in loc. cit. to clarify through Section 2 and Section 3,

• the relation between Gröbner bases in R and homogeneous Gröbner bases in $R[t]$ (Theorems 2.3 and 2.5), where $R = \oplus_{p\in\mathbb{N}} R_p$ is an $\mathbb{N}$-graded K-algebra with an SM K-basis (i.e., a skew multiplicative K-basis, see the definition in Section 1) consisting of homogeneous elements such that R has a Gröbner basis theory, and $R[t]$ is the polynomial ring in commuting variable t over R;

• and the relation between Gröbner bases in $K\langle X\rangle$ and homogeneous Gröbner bases in $K\langle X, T\rangle$ (Theorems 3.3 and 3.5), where $K\langle X\rangle = K\langle X_1, \dots, X_n\rangle$ is the free K-algebra of n generators and $K\langle X, T\rangle = K\langle X_1, \dots, X_n, T\rangle$ is the fee K-algebra of $n+1$ generators.

This makes a solid foundation for us to achieve the following goals: Firstly, in both cases mentioned above, demonstrate a general algorithmic principle of obtaining a Gröbner basis for an ideal I generated by non-homogeneous elements, and thereby obtaining a homogeneous Gröbner basis for the homogenization ideal of I, by passing to dealing with the homogenized generators (Propositions 2.7 and 3.7); Secondly, find all homogeneous Gröbner bases in $R[t]$ that correspond bijectively to all Gröbner bases in R (Theorem 2.9), respectively, find all homogeneous Gröbner bases in $K\langle X, T\rangle$ that correspond bijectively to all Gröbner bases in $K\langle X\rangle$ (Theorem 3.9); Thirdly, characterize all graded ideals in $R[t]$ that correspond bijectively to all ideals in R, respectively, characterize all graded ideals in $K\langle X, T\rangle$ that correspond bijectively to all ideals in $K\langle X\rangle$, in terms of Gröbner bases (Theorems 2.10 and 3.10). Based on the results obtained in previous sections, we show in Section 4 that algebras defined by dh-closed homogeneous Gröbner bases (see the definition in Sections 2 and 3) can be studied as Rees algebras (defined by grading filtration) effectively via studying algebras with simpler defining relations as demonstrated in ([10], [11]). A typical stage that may bring Propositions 4.1 and 4.2 into play is indicated by Theorem 2.12 (where $R[t]$ is replaced by the commutative polynomial K-algebra $K[x_1, \dots, x_n]$ with $R = K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n], t = x_i, 1 \le i \le n)$ and Theorem 3.12 (where $K\langle X, T\rangle = K\langle X_1, \dots, X_n\rangle$ with $K\langle X\rangle = K \langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\rangle, T = X_i, 1 \le i \le n)$.

Throughout the paper, $\mathbb{N}$ denotes the additive monoid of nonnegative integers; all algebras considered are associative algebras with multiplicative identity 1; and all ideals considered are meant two-sided ideals. If S is a nonempty subset of an algebra, then we use $\langle S\rangle$ to denote the two-sided ideal generated by S. Moreover, if K is a field, then we write K^* for the set of nonzero elements of K, i.e., $K^* = K - \{0\}$.

1. Gröbner Bases w.r.t. SM *K*-bases

Let K be a field. In this section, we sketch the Gröbner basis theory for K-algebras with an SM K-basis (i.e., a skew multiplicative K-basis) introduced in [10].

Let R be a K-algebra, and let $\mathcal{B}$ be a K-basis of R. Adopting the commonly used notation and terminology in computational algebra, we use lowercase letters $w, u, v, s, \dots$ to denote elements in $\mathcal{B}$ and call an element $w \in \mathcal{B}$ a *monomial*. If $\prec$ is a well-ordering on $\mathcal{B}$, $f \in R$, and

$$f = \sum_{i=1}^{s} \lambda_i u_i, \quad \lambda_i \in K^*, u_i \in \mathcal{B}, u_1 \prec u_2 \prec \cdots \prec u_s,$$

then the *leading monomial*, *leading coefficient*, and *leading term* of f are respectively, denoted by

$$\mathbf{LM}(f) = u_s, \quad \mathbf{LC}(f) = \lambda_s, \quad \mathbf{LT}(f) = \lambda_s u_s.$$

Furthermore, if $\prec$ satisfies the following conditions:

(Mo1) if $w, u, v \in \mathcal{B}$, $u \prec v$, and $uw \neq 0$, $vw \neq 0$, then $\mathbf{LM}(uw) \prec \mathbf{LM}(vw)$;

(Mo2) if $w, u, v \in \mathcal{B}$, $u \prec v$, and $wu \neq 0$, $wv \neq 0$, then $\mathbf{LM}(wu) \prec \mathbf{LM}(wv)$;

(Mo3) if $w, u, v \in \mathcal{B}$ and $\mathbf{LM}(uw) = v$, then $u \prec v$, $w \prec v$,

then $\prec$ is called a *monomial ordering* on $\mathcal{B}$.

If R has a K-basis $\mathcal{B}$ satisfying

$$u, v \in \mathcal{B} \text{ implies } \begin{cases} u \cdot v = \lambda w, \text{ where } \lambda \in K^*, w \in \mathcal{B}, \\ \text{or } u \cdot v = 0, \end{cases}$$

then $\mathcal{B}$ is called a *skew multiplicative K-basis* (abbreviated SM K-basis). If a K-algebra R has an SM K-basis and a monomial ordering $\prec$ on $\mathcal{B}$, then the pair $(\mathcal{B}, \prec)$ is called an *admissible system* of R; in this case, the division of monomials in R is defined as follows: $u, v \in \mathcal{B}$, $u|v$, if and only if there is some $\lambda \in K^*$ and $w, s \in \mathcal{B}$ such that $v = \lambda wus$; furthermore, the division of monomials induces a $\prec$-compatible division algorithm for elements in R, and consequently, a Gröbner basis theory for R may be carried out, that is, if I is a nonzero ideal of R, then I has a (finite or infinite) *Gröbner basis* $\mathcal{G}$ in the sense that if $f \in I$, $f \neq 0$, then there is some $g \in \mathcal{G}$ such that $\mathbf{LM}(g)|\mathbf{LM}(f)$ (see Proposition 1.2 below).

Commutative polynomial K-algebra, non-commutative free K-algebra, path algebra over K, the coordinate algebra of a quantum affine n-space over K, and exterior K-algebra are typical K-algebras with an SM K-basis and a Gröbner basis theory (cf. [1], [2], [3], [4], [5], [12]).

Theorem 1.1. *Suppose that the K-algebra R has an admissible system* $(\mathcal{B}, \prec)$ *with* $\mathcal{B}$ *an SM K-basis, and let I be an ideal of R. For a subset* $\mathcal{G} \subset I$, *the following statements are equivalent*:

(i) $\mathcal{G}$ *is a Gröbner basis of I*;

(ii) *For* $f \in I$, *if* $f \neq 0$, *then f has a Gröbner presentation, i.e.*,

$$f = \sum_{i,j} \lambda_{ij} w_{ij} g_j v_{ij}, \ \lambda_{ij} \in K^*, \ w_{ij}, v_{ij} \in \mathcal{B}, \ g_j \in \mathcal{G},$$

satisfying $\mathbf{LM}(w_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f)$, *and there is some* j^* *such that* $\mathbf{LM}(w_{ij^*} g_{j^*} v_{ij^*}) = \mathbf{LM}(f)$;

(iii) $\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle$, *where* $\langle \mathbf{LM}(I) \rangle$ *is the ideal of R generated by the set* $\mathbf{LM}(I) = \{\mathbf{LM}(f) \mid f \in I\}$, *and* $\langle \mathbf{LM}(\mathcal{G}) \rangle$ *is the ideal of R generated by the set* $\mathbf{LM}(\mathcal{G}) = \{\mathbf{LM}(f) \mid f \in \mathcal{G}\}$. □

Suppose that the *K*-algebra *R* has an admissible system $(\mathcal{B}, \prec)$ with $\mathcal{B}$ an SM *K*-basis, and let *I* be an ideal of *R*. If $\mathcal{G}$ is a Gröbner basis of *I* and any proper subset of $\mathcal{G}$ cannot be a Gröbner basis, then $\mathcal{G}$ is called a *minimal Gröbner basis* of *I*. It follows easily from Theorem 1.1(ii) that $\mathcal{G}$ is a minimal Gröbner basis of *I*, if and only if $\mathbf{LM}(g_1) \nmid \mathbf{LM}(g_2)$ for $g_1, g_2 \in \mathcal{G}$ with $g_1 \neq g_2$.

Proposition 1.2. *Suppose that the K-algebra R has an SM K-basis and an admissible system* $(\mathcal{B}, \prec)$. *The following two statements hold.*

(i) *Every ideal I of R has a minimal Gröbner basis*:

$$\mathcal{G} = \{g \in I \mid \text{ if } g' \in I \text{ and } g' \neq g, \text{ then } \mathbf{LM}(g') \nmid \mathbf{LM}(g)\}.$$

(ii) *If* $R = \oplus_{p \in \mathbb{N}} R_p$ *is an* $\mathbb{N}$-*graded K-algebra and* $\mathcal{B}$ *consists of* $\mathbb{N}$-*homogeneous elements*, *then every graded ideal I has a minimal homogeneous Gröbner basis*, *i.e.*, *a minimal Gröbner basis consisting of homogeneous elements* (*it is sufficient to consider homogeneous elements of I in* (i) *above*). □

By the definition of a minimal Gröbner basis, it is not difficult to see that, if $\mathcal{G}$ is any Gröbner basis of the ideal *I*, then the division algorithm enables us to produce from $\mathcal{G}$, a minimal Gröbner basis.

Let R be a K-algebra that has an SM K-basis $\mathcal{B}$ and an admissible system $(\mathcal{B}, \prec)$. Then, any nonempty subset $S \subset R$ determines a subset of monomials

$$N(S) = \{w \in \mathcal{B} \mid \mathbf{LM}(f) \nmid w,\ f \in S\},$$

which is usually called the set of *normal monomials* in $\mathcal{B}$ (mod S). Let I be an ideal of R. By Theorem 1.1, it is easy to see that, if $\mathcal{G}$ is a Gröbner basis of I, then $N(I) = N(\mathcal{G})$; and furthermore, each $f \in R$ has an expression of the form $f = \sum_{i,j} \lambda_{ij} w_{ij} g_j v_{ij} + r_f$, where $\lambda_{ij} \in K^*$, w_{ij}, $v_{ij} \in \mathcal{B}$, $g_j \in \mathcal{G}$, and either $r_f = 0$ or r_f has a unique linear expression of the form $r_f = \sum_\ell \lambda_\ell w_\ell$ with $\lambda_\ell \in K^*$, $w_\ell \in N(\mathcal{G})$.

We finish this section by characterizing a Gröbner basis $\mathcal{G}$ in terms of $N(\mathcal{G})$, which, in turn, gives rise to the fundamental decomposition of the K-space R by I, respectively by $\langle \mathbf{LM}(I) \rangle$.

Theorem 1.3. *Let $I = \langle \mathcal{G} \rangle$ be an ideal of R generated by the subset $\mathcal{G}$. With notation as above, the following statements are equivalent.*

(i) *$\mathcal{G}$ is a Gröbner basis of I.*

(ii) *Consider the K-subspace spanned by $N(\mathcal{G})$, denoted K-span $N(\mathcal{G})$. Then the K-space R has the decomposition*

$$R = I \oplus K\text{-}spanN(\mathcal{G}) = \langle \mathbf{LM}(I) \rangle \oplus K\text{-}spanN(\mathcal{G}).$$

(iii) *The canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $K\langle X \rangle / \langle \mathbf{LM}(I) \rangle$ and $K\langle X \rangle / I$ forms a K-basis for $K\langle X \rangle / \langle \mathbf{LM}(I) \rangle$ and $K\langle X \rangle / I$, respectively.*

2. Central (De)homogenized Gröbner Bases

Let $R = \oplus_{p \in \mathbb{N}} R_p$ be an $\mathbb{N}$-graded algebra over a field K. Throughout this section, we fix the following assumption: R has an SM K-basis $\mathcal{B}$ consisting of $\mathbb{N}$-homogeneous elements, i.e., if $w \in \mathcal{B}$, then $w \in R_p$ for some $p \in \mathbb{N}$.

By Section 1, if R has an admissible system $(\mathcal{B}, \prec)$, then every ideal of R has a Gröbner basis. Let $R[t]$ be the polynomial ring in commuting variable t over R (i.e., $rt = tr$ for all $r \in R$). Then, as we will see soon, with respect to the mixed $\mathbb{N}$-gradation and a suitable monomial ordering, $R[t]$ has a Gröbner basis theory, in particular, every graded ideal of $R[t]$ has a homogeneous Gröbner basis. By means of the central (de)homogenization technique as used in ([7], [9], [10]), the present section aims to clarify in detail the relation between Gröbner bases in R and homogeneous Gröbner bases in $R[t]$. Moreover, graded ideals in $R[t]$ that correspond bijectively to all ideals in R are characterized in terms of dh-closed Gröbner bases (see the definition later).

Note that $R[t]$ has the mixed $\mathbb{N}$-gradation, that is, $R[t] = \oplus_{p\in\mathbb{Z}} R[t]_p$ is an $\mathbb{N}$-graded algebra with the degree-p homogeneous part

$$R[t]_p = \left\{ \sum_{i+j=p} F_i t^j \;\middle|\; F_i \in R_i,\ j \geq 0 \right\},\ p \in \mathbb{N}.$$

Considering the onto ring homomorphism $\phi: R[t] \to R$ defined by $\phi(t) = 1$, then for each $f \in R$, there exists a homogeneous element $F \in R[t]_p$, for some p, such that $\phi(F) = f$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$, and $f_p \neq 0$, then

$$f^* = f_p + tf_{p-1} + \cdots + t^s f_{p-s}$$

is a homogeneous element of degree p in $R[t]_p$ satisfying $\phi(f^*) = f$. We call the homogeneous element f^* obtained this way the *central homogenization* of f with respect to t (for the reason that t is in the center of $R[t]$). On the other hand, for an element $F \in R[t]$, we write

$$F_* = \phi(F),$$

and call it the *central dehomogenization* of F with respect to t (again for the reason that t is in the center of $R[t]$). Hence, if I is an ideal R, then we write $I^* = \{f^* \mid f \in I\}$ and call the $\mathbb{N}$-graded ideal $\langle I^* \rangle$ generated by I^*, the *central homogenization ideal* of I in $R[t]$ with respect to t; and if J is an ideal of $R[t]$, then since ϕ is a ring epimorphism, $\phi(J)$ is an ideal of R, so we write J_* for $\phi(J) = \{H_* = \phi(H) \mid H \in J\}$ and call it the *central dehomogenization ideal* of J in R with respect to t. Consequently, henceforth, we will also use the notation $(J_*)^* = \{(h_*)^* \mid h \in J\}$.

Lemma 2.1. *With every definition and notation made above, the following statements hold.*

(i) *For* $F, G \in R[t]$, $(F + G)_* = F_* + G_*$, $(FG)_* = F_* G_*$.

(ii) *For any* $f \in R$, $(f^*)_* = f$.

(iii) *If* $F \in R[t]_p$ *and if* $(F_*)^* \in R[t]_q$, *then* $p \geq q$ *and* $t^r (F_*)^* = F$ *with* $r = p - q$.

(iv) *If I is an ideal of R, then each homogeneous element* $F \in \langle I^* \rangle$ *is of the form* $t^r f^*$, *for some* $r \in \mathbb{N}$ *and* $f \in I$.

(v) *If J is a graded ideal of* $R[t]$, *then for each* $h \in J_*$, *there is some homogeneous element* $F \in J$ *such that* $F_* = h$.

Proof. By the definition of central (de)homogenization, the verification of (i)-(v) is straight forward. □

Suppose that the given $\mathbb{N}$-graded K-algebra $R = \oplus_{p \in \mathbb{N}} R_p$ has an admissible system $(\mathcal{B}, \prec_{gr})$ with $\prec_{gr}$ an $\mathbb{N}$-*graded monomial ordering* on $\mathcal{B}$, i.e., the monomial ordering $\prec_{gr}$ is determined by a well-ordering $\prec$ on $\mathcal{B}$ subject to the rule: for $u, v \in \mathcal{B}$,

$$u \prec_{gr} v \text{ if } \begin{cases} \deg(u) < \deg(v), \\ \text{or} \\ \deg(u) = \deg(v) \text{ and } u \prec v, \end{cases}$$

where $\deg(\ )$ denotes the degree-function on elements of R (note that elements in $\mathcal{B}$ are homogeneous by our assumption). Taking the K-basis $\mathcal{B}^* = \{t^r w \mid w \in \mathcal{B},\ r \in \mathbb{N}\}$ of $R[t]$ into account, then since $\mathcal{B}^*$ is obviously a skew multiplicative K-basis for $R[t]$, the $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on $\mathcal{B}$ extends to a monomial ordering on $\mathcal{B}^*$, denoted $\prec_{t-gr}$, as follows:

$$t^{r_1} w_1 \prec_{t-gr} t^{r_2} w_2, \text{ if and only if } w_1 \prec_{gr} w_2, \text{ or } w_1 = w_2 \text{ and } r_1 < r_2.$$

Thus, $R[t]$ holds a Gröbner basis theory with respect to the admissible system $(\mathcal{B}^*, \prec_{t-gr})$.

It follows from the definition of $\prec_{t-gr}$ that $t^r \prec_{t-gr} w$ for all integers $r > 0$ and all $w \in \mathcal{B} - \{1\}$ (if $\mathcal{B}$ contains the identity element 1 of R). Hence, although elements of $\mathcal{B}^*$ are homogeneous with respect to the mixed $\mathbb{N}$-gradation of $R[t]$, $\prec_{t-gr}$ *is not a graded monomial ordering* on $\mathcal{B}^*$. Nevertheless, as described in the lemma below, since $\prec_{gr}$ is an $\mathbb{N}$-graded monomial ordering on $\mathcal{B}$, leading monomials with respect to both monomial orderings behave in a compatible way under taking the central (de)homogenization.

Lemma 2.2. *With notation given above, the following statements hold.*

(i) *If $f \in R$, then*

$$\mathbf{LM}(f^*) = \mathbf{LM}(f) \text{ w.r.t. } \prec_{t-gr} \text{ on } \mathcal{B}^*.$$

(ii) *If F is a nonzero homogeneous element of $R[t]$, then*

$$\mathbf{LM}(F_*) = \mathbf{LM}(F)_* \text{ w.r.t. } \prec_{gr} \text{ on } \mathcal{B}.$$

Proof. Since the central homogenization is done with respect to the degree of elements in R, that is, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$, and $f_p \neq 0$, then $f^* = f_p + tf_{p-1} + \cdots + t^s f_{p-s}$, the equality $\mathbf{LM}(f) = \mathbf{LM}(f_p) = \mathbf{LM}(f^*)$ follows immediately from the definitions of $\prec_{gr}$ and $\prec_{t-gr}$.

To prove (ii), let $F \in R[t]_p$ be a nonzero homogeneous element of degree p, say

$$F = \lambda t^r w + \lambda_1 t^{r_1} w_1 + \cdots + \lambda_s t^{r_s} w_s,$$

where $\lambda, \lambda_i \in K^*$, $r, r_i \in \mathbb{N}$, $w, w_i \in \mathcal{B}$, such that $\mathbf{LM}(F) = t^r w$. Noticing that $\mathcal{B}$ consists of $\mathbb{N}$-homogeneous elements and $R[t]$ has the mixed $\mathbb{N}$-gradation by the previously fixed assumption, we have $d(t^r w) = d(t^{r_i} w_i) = p$, $1 \le i \le s$. Thus, $w = w_i$ will imply $r = r_i$, and thereby $t^r w = t^{r_i} w_i$. So, we may assume that $w \ne w_i$, $1 \le i \le s$. Then, it follows from the definition of $\prec_{t-gr}$ that $w_i \prec_{gr} w$ and $r \le r_i$, $1 \le i \le s$. Therefore, $\mathbf{LM}(F_*) = w = \mathbf{LM}(F)_*$, as desired. □

The next result is a generalization of ([7] Theorem 2.3.2).

Theorem 2.3. *With notions and notations as fixed before, let I be an ideal of R, and $\langle I^* \rangle$ the central homogenization ideal of I in $R[t]$ with respect to t. For a subset $\mathcal{G} \subset I$, the following two statements are equivalent.*

(i) *$\mathcal{G}$ is a Gröbner basis for I in R with respect to the admissible system $(\mathcal{B}, \prec_{gr})$;*

(ii) *$\mathcal{G}^* = \{g^* \mid g \in \mathcal{G}\}$ is a Gröbner basis for $\langle I^* \rangle$ in $R[t]$ with respect to the admissible system $(\mathcal{B}^*, \prec_{t-gr})$.*

Proof. In proving the equivalence below, without specific indication, we shall use Lemma 2.2(i) wherever it is needed.

(i) $\Rightarrow$ (ii) First note that $\mathcal{G}^* \subset \langle I^* \rangle$. We prove further that, if $F \in \langle I^* \rangle$, then $\mathbf{LM}(g^*) | \mathbf{LM}(F)$ for some $g \in \mathcal{G}$. Since $\langle I^* \rangle$ is a graded ideal, we may assume, without loss of generality, that F is a homogeneous element. So, by Lemma 2.1(iv), we have $F = t^r f^*$ for some $f \in I$. It follows from the equality $\mathbf{LM}(f^*) = \mathbf{LM}(f)$ that

$$\mathbf{LM}(F) = t^r\mathbf{LM}(f^*) = t^r\mathbf{LM}(f). \tag{1}$$

If $\mathcal{G}$ is a Gröbner basis for I, then $\mathbf{LM}(f) = \lambda v\mathbf{LM}(g)w,$ for some $\lambda \in K^*$, $g \in \mathcal{G}$, and $v, w \in \mathcal{B}$. Thus, by (1) above, we get

$$\mathbf{LM}(F) = t^r\mathbf{LM}(f) = \lambda t^r v\mathbf{LM}(g^*)w.$$

This shows that $\mathbf{LM}(g^*) \mid \mathbf{LM}(F)$, as desired.

(ii) $\Rightarrow$ (i) Suppose $\mathcal{G}^*$ is a Gröbner basis for the homogenization ideal $\langle I^* \rangle$ of I in $R[t]$. Let $f \in I$. Then $\mathbf{LM}(f) = \mathbf{LM}(f^*) = \lambda v\mathbf{LM}(g^*)w$ $= \lambda v\,\mathbf{LM}(g)w$, for some $\lambda \in K^*$, $v, w \in \mathcal{B}$, and $g \in \mathcal{G}$. This shows that $\mathbf{LM}(g) \mid \mathbf{LM}(f)$, i.e., $\mathcal{G}$ is a Gröbner basis for I in R. □

We call the Gröbner basis $\mathcal{G}^*$ obtained in Theorem 2.3 the *central homogenization* of $\mathcal{G}$ in $R[t]$ with respect to t, or $\mathcal{G}^*$ is a *central homogenized Gröbner basis* with respect to t.

By Theorem 1.3, Lemma 2.2, and Theorem 2.3, we have immediately the following corollary.

Corollary 2.4. *Let I be an arbitrary ideal of R. With notation as before, if $\mathcal{G}$ is a Gröbner basis of I with respect to the data $(\mathcal{B}, \prec_{gr})$, then, with respect to the data $(\mathcal{B}^*, \prec_{t-gr})$, we have*

$$N(\langle I^* \rangle) = N(\mathcal{G}^*) = \{t^r w \mid w \in N(\mathcal{G}),\ r \in \mathbb{N}\},$$

that is, the set $N(\langle I^* \rangle)$ of normal monomials in $\mathcal{B}^*$ $(\bmod\ \langle I^* \rangle)$ is determined by the set $N(I) = N(\mathcal{G})$ of normal monomials in $\mathcal{B}$ (mod I). Hence, the algebra $R[t]/\langle I^* \rangle = R[t]/\langle \mathcal{G}^* \rangle$ has the K-basis

$$\overline{N(\langle I^* \rangle)} = \{\overline{t^r w} \mid w \in N(I),\ r \in \mathbb{N}\}. \qquad \square$$

We may also obtain a Gröbner basis for an ideal I of R by dehomogenizing a homogeneous Gröbner basis of the ideal $\langle I^* \rangle \subset R[t]$. Below, we give a more general approach to this assertion.

Theorem 2.5. *Let J be a graded ideal of $R[t]$. If $\mathcal{G}$ is a homogeneous Gröbner basis of J with respect to the data $(\mathcal{B}^*, \prec_{t-gr})$, then $\mathcal{G}_* = \{G_* \mid G \in \mathcal{G}\}$ is a Gröbner basis for the ideal J_* in R with respect to the data $(\mathcal{B}, \prec_{gr})$.*

Proof. If $\mathcal{G}$ is a Gröbner basis of J, then $\mathcal{G}$ generates J and hence $\mathcal{G}_* = \phi(\mathcal{G})$ generates $J_* = \phi(J)$. For a nonzero $f \in J_*$, by Lemma 2.1(v), there exists a homogeneous element $H \in J$ such that $H_* = f$. It follows from Lemma 2.2 that

$$\mathbf{LM}(f) = \mathbf{LM}(f^*) = \mathbf{LM}((H_*)^*). \qquad (1)$$

On the other hand, there exists some $G \in \mathcal{G}$ such that $\mathbf{LM}(G) \mid \mathbf{LM}(H)$, i.e.,

$$\mathbf{LM}(H) = \lambda t^{r_1} w \mathbf{LM}(G) t^{r_2} v, \qquad (2)$$

for some $\lambda \in K^*$, $r_1, r_2 \in \mathbb{N}$, $w, v \in \mathcal{B}$. But by Lemma 2.1(iii), we also have $t^r (H_*)^* = H$ for some $r \in \mathbb{N}$, and hence

$$\mathbf{LM}(H) = \mathbf{LM}(t^r (H_*)^*) = t^r \mathbf{LM}(H_*)^*). \qquad (3)$$

So, (1) + (2) + (3) yields

$$\begin{aligned} \lambda t^{r_1 + r_2} w \mathbf{LM}(G) v &= \mathbf{LM}(H) \\ &= t^r \mathbf{LM}((H_*)^*) \\ &= t^r \mathbf{LM}(f). \end{aligned}$$

Dehomogenizing both sides of the above equality, by Lemmas 2.1(i) and 2.2(ii), we obtain

$$\lambda w \mathbf{LM}(G_*)v = \lambda w \mathbf{LM}(G)_* v = \mathbf{LM}(f).$$

This shows that $\mathbf{LM}(G_*) | \mathbf{LM}(f)$. Therefore, $\mathscr{G}_*$ is a Gröbner basis for J_*. □

We call the Gröbner basis $\mathscr{G}_*$ obtained in Theorem 2.5, the *central dehomogenization of* $\mathscr{G}$ in R with respect to t, or $\mathscr{G}_*$ is a *central dehomogenized Gröbner basis* with respect to t.

Corollary 2.6. *Let I be an ideal of R. If $\mathscr{G}$ is a homogeneous Gröbner basis of $\langle I^* \rangle$ in $R[t]$ with respect to the data $(\mathcal{B}^*, \prec_{t-gr})$, then $\mathscr{G}_* = \{g_* | \, g \in \mathscr{G}\}$ is a Gröbner basis for I in R with respect to the data $(\mathcal{B}, \prec_{gr})$. Moreover, if I is generated by the subset F and $F^* \subset \mathscr{G}$, then $F \subset \mathscr{G}_*$.*

Proof. Put $J = \langle I^* \rangle$. Then since $J_* = I$, it follows from Theorem 1.5 that, if $\mathscr{G}$ is a homogeneous Gröbner basis of J, then $\mathscr{G}_*$ is a Gröbner basis for I. The second assertion of the theorem is clear by Lemma 1.1(ii).

□

Let S be a nonempty subset of R and $I = \langle S \rangle$, the ideal generated by S. Then, with $S^* = \{f^* \,|\, f \in S\}$, in general $\langle S^* \rangle \subsetneq \langle I^* \rangle$ in $R[t]$ (for instance, consider $S = \{y^3 - x - y, \, y^2 + 1\}$ in the commutative polynomial ring $K[x, y]$ and S^* in $K[x, y, t]$ with respect to t). So, from both a practical and a computational viewpoint, it is the right place to set up the procedure of getting a Gröbner basis for I, and hence a Gröbner basis for $\langle I^* \rangle$ by producing a homogeneous Gröbner basis of the graded ideal $\langle S^* \rangle$.

Proposition 2.7. *Let $I = \langle S \rangle$ be the ideal of R generated by a subset S. Suppose that Gröbner bases are algorithmically computable in R and hence in $R[t]$. Then a Gröbner basis for I and a homogeneous Gröbner basis for $\langle I^* \rangle$ may be obtained by implementing the following procedure*:

Step 1. *Starting with the initial subset* $S^* = \{f^* \mid f \in S\}$ *consisting of homogeneous elements, compute a homogeneous Gröbner basis* $\mathscr{G}$ *for the graded ideal* $\langle S^* \rangle$ *of* $R[t]$.

Step 2. *Noticing* $\langle S^* \rangle_* = I$, *use Theorem* 2.5 *and dehomogenize* $\mathscr{G}$ *with respect to t in order to obtain the Gröbner basis* $\mathscr{G}_*$ *for I.*

Step 3. *Use Theorem* 2.3 *and homogenize* $\mathscr{G}_*$ *with respect to t in order to obtain the homogeneous Gröbner basis* $(\mathscr{G}_*)^*$ *for the graded ideal* $\langle I^* \rangle$.

□

Based on Theorems 2.3 and 2.5, we proceed now to find those homogeneous Gröbner bases in $R[t]$ that correspond bijectively to all Gröbner bases in R.

Considering the central (de)homogenization with respect to t, a homogeneous element $F \in R[t]$ is called *dh-closed* if $(F_*)^* = F$; a subset S of $R[t]$ consisting of dh-closed homogeneous elements is called a *dh-closed homogeneous set*; if a dh-closed homogeneous set $\mathscr{G}$ in $R[t]$ forms a Gröbner basis with respect to $\prec_{t-gr}$, then it is called a *dh-closed homogeneous Gröbner basis*.

To better understand the dh-closed property introduced above, we characterize a dh-closed homogeneous element as follows.

Lemma 2.8. *With notation as before, for a homogeneous element* $F \in R[t]$, *with respect to* $(\mathcal{B}, \prec_{gr})$ *and* $(\mathcal{B}^*, \prec_{t-gr})$, *the following statements are equivalent*:

(i) *F is dh-closed, i.e.,* $(F_*)^* = F$;

(ii) $\mathbf{LM}(F_*) = \mathbf{LM}(F)$;

(iii) *F cannot be written as* $F = t^r H$ *with H a homogeneous element of* $R[t]$ *and* $r \geq 1$;

(iv) $t \nmid \mathbf{LM}(F)$.

Proof. Using Lemma 2.1 combined with the definitions of $\prec_{t-gr}$ and $(F_*)^*$, the verification of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) is straightforward. □

Theorem 2.9. *With respect to the systems* $(\mathcal{B}, \prec_{gr})$ *and* $(\mathcal{B}^*, \prec_{t-gr})$, *there is an one-to-one correspondence between the set of all Gröbner bases in R and the set of all dh-closed homogeneous Gröbner bases in* $R[t]$:

$$\begin{array}{ccc} \{\text{Gröbner bases } \mathcal{G} \text{ in } R\} & \leftrightarrow & \left\{\begin{array}{l}\text{dh-closed homogeneous} \\ \text{Gröbner bases } \mathscr{G} \text{ in } R[t]\end{array}\right\}, \\ \mathcal{G} & \to & \mathcal{G}^*, \\ \mathscr{G}_* & \leftarrow & \mathscr{G}, \end{array}$$

and this correspondence also gives rise to a bijective map between the set of all minimal Gröbner bases in R and the set of all dh-closed minimal homogeneous Gröbner bases in $R[t]$.

Proof. Bearing the definitions of homogenization and dehomogenization with respect to t in mind, by Theorems 2.3 and 2.5, it can be verified directly that the given rule of correspondence defines an one-to-one map. By the definition of a minimal Gröbner basis, the second assertion follows from Lemmas 2.2(i) and 2.8(ii). □

Below, we characterize the graded ideal generated by a dh-closed homogeneous Gröbner basis in $R[t]$.

Theorem 2.10. *With notation as before, let J be a graded ideal of* $R[t]$ *and* $\mathscr{G}$ *be a minimal homogeneous Gröbner basis of J. Under* $(\mathcal{B}, \prec_{gr})$ *and* $(\mathcal{B}^*, \prec_{t-gr})$, *the following statements are equivalent*:

(i) $\mathscr{G}$ *is a dh-closed homogeneous Gröbner basis*;

(ii) *J has the property* $\langle (J_*)^* \rangle = J$;

(iii) *The* $R[t]$-*module* $R[t]/J$ *is t-torsionfree, i.e., if* $\bar{f} = f + J \in R[t]/J$ *and* $\bar{f} \neq 0$, *then* $t\bar{f} \neq 0$, *or equivalently*, $tf \notin J$;

(iv) $tR[t] \cap J = tJ$.

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.5, $\mathscr{G}_*$ is a Gröbner basis for J_* in R with respect to $(\mathcal{B}, \prec_{gr})$. Since $\mathscr{G}$ is dh-closed, it follows from Theorem 2.3 that, $\mathscr{G} = (\mathscr{G}_*)^*$ is a Gröbner basis for $\langle (J_*)^* \rangle$. This shows that $J = \langle \mathscr{G} \rangle = \langle (J_*)^* \rangle$.

(ii) $\Rightarrow$ (iii) Note that J is a graded ideal and t is a homogeneous element in $R[t]$. It is sufficient to prove that t does not annihilate any nonzero homogeneous element of $R[t]/J$. Thus, assuming $F \in R[t]_p$ and $tF \in J$, then since $J = \langle (I_*)^* \rangle$, we have

$$(F_*)^* = ((tF_*)^* \in \langle (J_*)^* \rangle = J.$$

It follows from Lemma 2.1(iii) that, there exists some $r \in \mathbb{N}$ such that $F = t^r (F_*)^* \in J$, as desired.

(iii) $\Leftrightarrow$ (iv) Obvious.

(iii) $\Rightarrow$ (i) Note that $\mathscr{G}$ is a homogeneous Gröbner basis by the assumption. For each $g \in \mathscr{G}$, by Lemma 2.1(iii), there is some $r \in \mathbb{N}$ such that $t^r (g_*)^* = g$. It follows that with respect to $\prec_{t-gr}$, we have $\mathbf{LM}(g) = t^r \mathbf{LM}((g_*)^*)$. Since $R[t]/J$ is t-torsionfree, if $r > 0$, then $(g_*)^* \in J$. Thus, there is some $g' \in \mathscr{G}$ such that $g' \neq g$ and $\mathbf{LM}(g') | \mathbf{LM}((g_*)^*)$. Hence $\mathbf{LM}(g') | \mathbf{LM}(g)$, contradicting the assumption that $\mathscr{G}$ is a minimal Gröbner basis. Therefore, we must have $r = 0$, i.e., $(g_*)^* = g$. This shows that $\mathscr{G}$ is dh-closed. □

Corollary 2.11. *With notation as before, let J be a graded ideal of $R[t]$. If, with respect to $(\mathcal{B}, \prec_{gr})$ and $(\mathcal{B}^*, \prec_{t-gr})$, J has a dh-closed minimal homogeneous Gröbner basis, then every minimal homogeneous Gröbner basis of J is dh-closed.*

Proof. This follows from the fact that each of the properties (ii) – (iv) in Theorem 2.10 does not depend on the choice of the generating set for J. $\square$

Let J be a graded ideal of $R[t]$. If J has the property mentioned in Theorem 2.10(ii), i.e., $\langle (J_*)^* \rangle = J$, then we call J a *dh-closed graded ideal*. This definition generalizes the notion of a $(\phi_*)^*$-closed graded ideal introduced in ([8], CH.III). It is easy to see that, there is an one-to-one correspondence between the set of all ideals in R and the set of all dh-closed graded ideals in $R[t]$:

$$\begin{array}{ccc} \{\text{ideals } I \text{ in } R\} & \leftrightarrow & \{\text{dh-closed graded ideals } J \text{ in } R[t]\}, \\ I & \rightarrow & \langle I^* \rangle, \\ J_* & \leftarrow & J. \end{array}$$

By the foregoing argument, to know whether a given graded ideal J of $R[t]$ is dh-closed, it is sufficient to compute a minimal homogeneous Gröbner basis $\mathscr{G}$ for J (if Gröbner basis is computable in R), and then use the definition of a dh-closed homogeneous set or Lemma 2.8 to check whether $\mathscr{G}$ is dh-closed. This procedure may be realized in, for instance, commutative polynomial K-algebras, non-commutative free K-algebras, path algebras over K, the coordinate algebra of a quantum affine n-space over K, and exterior K-algebras, because Gröbner bases are computable in these algebras and their polynomial extensions.

Focusing on the commutative polynomial K-algebra $K[x_1, \ldots, x_n]$ in n variables, the good thing is that, the foregoing results can be applied to $K[x_1, \ldots, x_n]$ with respect to each x_i, $1 \le i \le n$. To see this clearly, let us put $x_i = t$, $R = K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, and $K[x_1, \ldots, x_n] = R[t]$. Moreover, let $(\mathcal{B}, \prec_{gr})$ be any fixed admissible system for R, where $\prec_{gr}$ is a graded monomial ordering on the standard K-basis $\mathcal{B}$ of R with respect to a fixed (weight) $\mathbb{N}$-gradation. Then $R[t]$ has the mixed $\mathbb{N}$-gradation and the corresponding admissible system $(\mathcal{B}^*, \prec_{t-gr})$, where $\prec_{t-gr}$ is the monomial ordering obtained by extending $\prec_{gr}$ on the standard K-basis $\mathcal{B}^*$ of $R[t]$. Instead of mentioning a version of each result obtained before, we highlight the respective version of Theorems 2.5 and 2.9 in this case as follows.

Theorem 2.12. *With the preparation made above, the following statements hold.*

(i) *For each* $x_i = t, 1 \leq i \leq n,$ *if* $\mathscr{G}$ *is a homogeneous Gröbner basis of the graded ideal* J *in* $R[t] = K[x_1, \ldots, x_n]$ *with respect to* $(\mathcal{B}^*, \prec_{t-gr})$, *then* $\mathscr{G}_* = \{g_* \mid g \in \mathscr{G}\}$ *is a Gröbner basis for the ideal* J_* *in* $R = K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ *with respect to* $(\mathcal{B}, \prec_{gr})$.

(ii) *For each* $x_i = t, 1 \leq i \leq n,$ *there is an one-to-one correspondence between the set of all dh-closed homogeneous Gröbner bases in* $R[t] = K[x_1, \ldots, x_n]$ *and the set of all Gröbner bases in* $R = K[x_1, \ldots, x_{i-1}, x_{n+1}, \ldots, x_n]$, *under which dh-closed minimal Gröbner bases correspond to minimal Gröbner bases.* □

Geometrically, Theorem 2.12 may be viewed as a Gröbner basis realization of the correspondence between algebraic sets in the projective space $\mathbb{P}_K^{n-1}$ and algebraic sets in the affine space $\mathbb{A}_K^{n-1}$, where $n \geq 2$.

3. Noncentral (De)homogenized Gröbner Bases

In this section, we clarify in detail how Gröbner bases in the free K-algebra $K\langle X\rangle = K\langle X_1, \ldots X_n\rangle$ of n generators are related to homogeneous Gröbner bases in the free K-algebra $K\langle X, T\rangle = K\langle X_1, \ldots, X_n, T\rangle$ of $n+1$ generators, if the noncentral (de)homogenization with respect to T is employed. Moreover, in terms of dh-closed Gröbner bases (see the definition later), we characterize graded ideals of $K\langle X, T\rangle$ that correspond bijectively to all ideals in $K\langle X\rangle$.

For a general algorithmic Gröbner basis theory, we refer to [12].

Let $K\langle X\rangle$ be equipped with a fixed *weight* $\mathbb{N}$-*gradation*, say each X_i has degree $n_i > 0, 1 \leq i \leq n$. Assigning to T, the degree 1 in $K\langle X, T\rangle$

and using the same weight n_i for each X_i as in $K\langle X\rangle$, we get the weight $\mathbb{N}$-gradation of $K\langle X, T\rangle$, which extends the weight $\mathbb{N}$-gradation of $K\langle X\rangle$. Let $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ denote the standard K-bases of $K\langle X\rangle$ and $K\langle X, T\rangle$, respectively. To be convenient, we use lowercase letters $w, u, v, \ldots$ to denote monomials in $\mathcal{B}$ as before, but use capitals $W, U, V, \ldots$ to denote monomials in $\widetilde{\mathcal{B}}$.

In what follows, we fix an admissible system $(\mathcal{B}, \prec_{gr})$ for $K\langle X\rangle$, where $\prec_{gr}$ is an $\mathbb{N}$-*graded lexicographic ordering* on $\mathcal{B}$ with respect to the fixed weight $\mathbb{N}$-gradation of $K\langle X\rangle$, such that

$$X_{i_1} \prec_{gr} X_{i_2} \prec_{gr} \cdots \prec_{gr} X_{i_n}.$$

Then, it is not difficult to see that $\prec_{gr}$ can be extended to an $\mathbb{N}$-*graded lexicographic ordering* $\prec_{T-gr}$ on $\widetilde{\mathcal{B}}$ with respect to the fixed weight $\mathbb{N}$-gradation of $K\langle X, T\rangle$, such that

$$T \prec_{T-gr} X_{i_1} \prec_{T-gr} X_{i_2} \prec_{T-gr} \cdots \prec_{T-gr} X_{i_n},$$

and thus, we get the admissible system $(\widetilde{\mathcal{B}}, \prec_{T-gr})$ for $K\langle X, T\rangle$. With respect to $\prec_{gr}$ and $\prec_{T-gr}$, we use $\mathbf{LM}(\ )$ to denote taking the leading monomial of elements in $K\langle X\rangle$ and $K\langle X, T\rangle$, respectively.

Consider the fixed $\mathbb{N}$-graded structures $K\langle X\rangle = \oplus_{p\in\mathbb{N}} K\langle X\rangle_p$, $K\langle X, T\rangle = \oplus_{p\in\mathbb{N}} K\langle X, T\rangle_p$, and the ring epimorphism $\psi : K\langle X, T\rangle \to K\langle X\rangle$ defined by $\psi(X_i) = X_i$ and $\psi(T) = 1$. Then each $f \in K\langle X\rangle$ is the image of some homogeneous element in $K\langle X, T\rangle$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in K\langle X\rangle_p$, $f_{p-j} \in K\langle X\rangle_{p-j}$, and $f_p \neq 0$, then

$$\widetilde{f} = f_p + Tf_{p-1} + \cdots + T^s f_{p-s}$$

is a homogeneous element of degree p in $K\langle X, T\rangle_p$ such that $\psi(\widetilde{f}) = f.$ We call the homogeneous element $\widetilde{f}$ obtained this way the *noncentral homogenization* of f with respect to T (for the reason that T is not a commuting variable). On the other hand, for $F \in K\langle X, T\rangle,$ we write

$$F_{\sim} = \psi(F),$$

and call $F_{\sim}$ the *noncentral dehomogenization* of F with respect to T (again for the reason that T is not a commuting variable). Furthermore, if $I = \langle S\rangle$ is the ideal of $K\langle X\rangle$ generated by a subset S, then we define

$$\widetilde{S} = \{\widetilde{f} \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \le i \le n\},$$

$$\widetilde{I} = \{\widetilde{f} \mid f \in I\} \cup \{X_iT - TX_i \mid 1 \le i \le n\},$$

and call the graded ideal $\langle \widetilde{I}\rangle$ generated by $\widetilde{I}$, the *noncentral homogenization ideal* of I in $K\langle X, T\rangle$ with respect to T; while if J is an ideal of $K\langle X, T\rangle,$ then since ψ is a surjective ring homomorphism, $\psi(J)$ is an ideal of $K\langle X\rangle,$ so we write $J_{\sim}$ for $\psi(J) = \{H_{\sim} = \psi(H) \mid H \in J\}$ and call it the *noncentral dehomogenization ideal* of J in $K\langle X\rangle$ with respect to T. Consequently, henceforth, we will also use the notation

$$(J_{\sim})^{\sim} = \{(h_{\sim})^{\sim} \mid h \in J\} \cup \{X_iT - TX_i \mid 1 \le i \le n\}.$$

It is straightforward to check that, with resspect to the data $(\widetilde{\mathcal{B}}, \prec_{T-gr}),$ the subset $\{X_iT - TX_i \mid 1 \le i \le n\}$ of $K\langle X, T\rangle$ forms a homogeneous Gröbner basis with $\mathbf{LM}(X_iT - TX_i) = X_iT,\ 1 \le i \le n.$ In the later discussion, we will freely use this fact without extra indication.

Lemma 3.1. *With notation as fixed before, the following properties hold.*

(i) *If* $F, G \in K\langle X, T\rangle,$ *then* $(F + G)_{\sim} = F_{\sim} + G_{\sim},\ (FG)_{\sim} = F_{\sim}G_{\sim}.$

(ii) *For each nonzero* $f \in K\langle X\rangle,\ (\widetilde{f})_{\sim} = f.$

(iii) *Let* $\mathscr{C}$ *be the graded ideal of* $K\langle X, T\rangle$ *generated by* $\{X_iT - TX_i \mid 1 \le i \le n\}$. *If* $F \in K\langle X, T\rangle_p$, *then there exists an* $L \in \mathscr{C}$ *and a unique homogeneous element of the form* $H = \sum_i \lambda_i T^{r_i} w_i \in K\langle X, T\rangle_p$, *where* $\lambda_i \in K^*$, $w_i \in \mathcal{B}$, *such that* $F = L + H$; *moreover, there is some* $r \in \mathbb{N}$ *such that* $T^r(H_\sim)^\sim = H$, *and hence,* $F = L + T^r(F_\sim)^\sim$.

(iv) *Let* $\mathscr{C}$ *be as in* (iii) *above. If I is an ideal of* $K\langle X\rangle$, $F \in \langle \widetilde{I}\rangle$ *is a homogeneous element, then there exist some* $L \in \mathscr{C}$, $f \in I$, *and* $r \in \mathbb{N}$ *such that* $F = L + T^r\widetilde{f}$.

(v) *If* J *is a graded ideal of* $K\langle X, T\rangle$ *and* $\{X_iT - TX_i \mid 1 \le i \le n\} \subset J$, *then for each nonzero* $h \in J_\sim$, *there exists a homogeneous element* $H = \sum_i \lambda_i T^{r_i} w_i \in J$, *where* $\lambda_i \in K^*$, $r_i \in \mathbb{N}$, *and* $w_i \in \mathcal{B}$, *such that for some* $r \in \mathbb{N}$, $T^r(H_\sim)^\sim = H$ *and* $H_\sim = h$.

Proof. (i) and (ii) follow from the definitions of noncentral homogenization and noncentral dehomogenization directly.

(iii) Since the subset $\{X_iT - TX_i \mid 1 \le i \le n\}$ is a Gröbner basis in $K\langle X, T\rangle$ with respect to $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, such that $\mathbf{LM}(X_iT - TX_i) = X_iT$, $1 \le i \le n$, if $F \in K\langle X, T\rangle_p$, then the division of F by this subset yields $F = L + H$, where $L \in \mathscr{C}$, and $H = \sum_i \lambda_i T^{r_i} w_i$ is the unique remainder with $\lambda_i \in K^*$, $w_i \in \mathcal{B}$, in which each monomial $T^r w_i$ is of degree p. By the definition of $\prec_{T-gr}$, the definition of noncentral homogenization, and the definition of noncentral dehomogenization, it is not diffcult to see that H has the desired property.

(iv) By (iii), $F = L + T^r(F_\sim)^\sim$ with $L \in \mathscr{C}$ and $r \in \mathbb{N}$. Since by (ii), we have $F_\sim \in \langle \widetilde{I}\rangle_\sim = I$, thus $f = F_\sim$ is the desired element.

(v) Using basic properties of homogeneous element and graded ideal in a graded ring, this follows from the foregoing (iii). □

As in the case of using the central (de)homogenization, before turning to deal with Gröbner bases, we are also concerned about the behavior of leading monomials under taking the noncentral (de)homogenization.

Lemma 3.2. *With the assumptions and notations as fixed above, the following statements hold.*

(i) *For each nonzero* $F \in K\langle X\rangle$, *we have*

$$\mathbf{LM}(f) = \mathbf{LM}(\widetilde{f}\,)\ w.r.t.\ \prec_{T-gr}\ on\ \widetilde{\mathcal{B}}.$$

(ii) *If* F *is a homogeneous element in* $K\langle X, T\rangle$ *such that* $X_iT \nmid \mathbf{LM}(F)$ *with respect to* $\prec_{T-gr}$ *for all* $1 \le i \le n$, *then* $\mathbf{LM}(F) = T^r w$, *for some* $r \in \mathbb{N}$ *and* $w \in \mathcal{B}$, *such that*

$$\mathbf{LM}(F_{\sim}) = w = \mathbf{LM}(F)_{\sim} w.r.t.\ \prec_{gr}\ on\ \mathcal{B}.$$

Proof. Since the noncentral homogenization is done with respect to the degree of elements in $K\langle X\rangle$, that is, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in K\langle X\rangle_p, f_{p-j} \in K\langle X\rangle_{p-j}$, and $f_p \neq 0$, then $\widetilde{f} = f_p + Tf_{p-1} + \cdots + T^s$ f_{p-s}, the equality $\mathbf{LM}(f) = \mathbf{LM}(f_p) = \mathbf{LM}(\widetilde{f}\,)$ follows immediately from the definitions of $\prec_{gr}$ and $\prec_{t-gr}$.

To prove (ii), let $F \in K\langle X, T\rangle_p$ be a nonzero homogeneous element of degree p. Then by the assumption, F may be written as

$$F = \lambda T^r w + \lambda_1 T^{r_1} X_{j_1} W_1 + \lambda_2 T^{r_2} X_{j_2} W_2 + \cdots + \lambda_s T^{r_s} X_{j_s} W_s,$$

where $\lambda, \lambda_i \in K^*, r, r_i \in \mathbb{N}, w \in \mathcal{B}$, and $W_i \in \widetilde{\mathcal{B}}$, such that $\mathbf{LM}(F) = T^r w$. Since $\mathcal{B}$ consists of $\mathbb{N}$-homogeneous elements and the $\mathbb{N}$-gradation of $K\langle X\rangle$ extends to give the $\mathbb{N}$-gradation of $K\langle X, T\rangle$, we have $d(T^r w) = d(T^{r_i} X_{j_i} W_i) = p, 1 \le i \le s$. Also note that T has degree 1. Thus, $w =$

$X_{j_i}W_i$ will imply $r = r_i$ and thereby, $T^r w = T^{r_i} X_{j_i} W_i$. So, we may assume, without loss of generality, that $w \neq X_{j_i} W_i$, $1 \le i \le s$. Then, it follows from the definition of $\prec_{T-gr}$ that $r \le r_i$, $1 \le i \le n$. Hence, $X_{j_i} W_i \prec_{T-gr} w, 1 \le i \le s$. Therefore, $(X_{j_i} W_i)_{\sim} \prec_{gr} w, 1 \le i \le n$, and consequently, $\mathbf{LM}(F_{\sim}) = w = \mathbf{LM}(F)_{\sim}$, as desired. □

The next result strengthens ([7], Theorem 2.3.1) and ([10], Theorem 8.2), in particular, the proof of (i) $\Rightarrow$ (ii) given below improves the proof of the same deduction given in [10].

Theorem 3.3. *With the notions and notations as fixed above, let* $I = \langle \mathcal{G} \rangle$ *be the ideal of* $K\langle X \rangle$ *generated by a subset* $\mathcal{G}$, *and* $\langle \widetilde{I} \rangle$, *the noncentral homogenization ideal of I in* $K\langle X, T \rangle$ *with respect to T. The following two statements are equivalent.*

(i) $\mathcal{G}$ *is a Gröbner basis of I with respect to the admissible system* $(\mathcal{B}, \prec_{gr})$ *of* $K\langle X \rangle$;

(ii) $\widetilde{\mathcal{G}} = \{\widetilde{g} \mid g \in \mathcal{G}\} \cup \{X_i T - TX_i \mid 1 \le i \le n\}$ *is a homogeneous Gröbner basis for* $\langle \widetilde{I} \rangle$ *with respect to the admissible system* $(\widetilde{\mathcal{B}}, \prec_{T-gr})$ *of* $K\langle X, T \rangle$.

Proof. In proving the equivalence below, without specific indication, we shall use Lemma 3.2(i) wherever it is needed.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{G}$ is a Gröbner basis for I with respect to the data $(\mathcal{B}, \prec_{gr})$. Let $F \in \langle \widetilde{I} \rangle$. Then since $\langle \widetilde{I} \rangle$ is a graded ideal, we may assume, without loss of generality, that F is a nonzero homogeneous element. We want to show that, there is some $D \in \widetilde{\mathcal{G}}$ such that $\mathbf{LM}(D) \mid \mathbf{LM}(F)$, and hence $\widetilde{\mathcal{G}}$ is a Gröbner basis.

Note that $\{X_i T - TX_i \mid 1 \le i \le n\} \subset \widetilde{\mathcal{G}}$ with $\mathbf{LM}(X_i T - TX_i) = X_i T$. If $X_i T \mid \mathbf{LM}(F)$ for some $X_i T$, then we are done. Otherwise,

$X_iT \nmid \mathbf{LM}(F)$ for all $1 \le i \le n$. Thus, by Lemma 3.2(ii), $\mathbf{LM}(F) = T^r w$ for some $r \in \mathbb{N}$ and $w \in \mathcal{B}$, such that

$$\mathbf{LM}(F_{\sim}) = w = \mathbf{LM}(F)_{\sim}. \tag{1}$$

On the other hand, by Lemma 3.1(iv), we have $F = L + T^q \widetilde{f}$, where L is an element in the ideal $\mathscr{C}$ generated by $\{X_iT - TX_i \mid 1 \le i \le n\}$ in $K\langle X, T\rangle$, $q \in \mathbb{N}$, and $f \in I$. It turns out that

$$F_{\sim} = (\widetilde{f})_{\sim} = f, \text{ and hence } \mathbf{LM}(F_{\sim}) = \mathbf{LM}(f). \tag{2}$$

But, since $\mathcal{G}$ is a Gröbner basis for I, there is some $g \in \mathcal{G}$ such that $\mathbf{LM}(g) \mid \mathbf{LM}(f)$, i.e., there are $u, v \in \mathcal{B}$ such that

$$\mathbf{LM}(f) = u\mathbf{LM}(g)v = u\mathbf{LM}(\widetilde{g})v. \tag{3}$$

Combining (1), (2), and (3) above, we have

$$w = \mathbf{LM}(F_{\sim}) = \mathbf{LM}(f) = u\mathbf{LM}(\widetilde{g})v.$$

Therefore, $\mathbf{LM}(\widetilde{g}) \mid T^r w$, i.e., $\mathbf{LM}(\widetilde{g}) \mid \mathbf{LM}(F)$, as desired.

(ii) $\Rightarrow$ (i) Suppose that $\widetilde{\mathcal{G}}$ is a Gröbner basis of the graded ideal $\langle \widetilde{I} \rangle$ in $K\langle X, T\rangle$. If $f \in I$, then since $\widetilde{f} \in \widetilde{I}$, there is some $D \in \widetilde{\mathcal{G}}$ such that $\mathbf{LM}(D) \mid \mathbf{LM}(\widetilde{f})$. Note that $\mathbf{LM}(\widetilde{f}) = \mathbf{LM}(f)$ and thus $T \nmid \mathbf{LM}(\widetilde{f})$. Hence $D = \widetilde{g}$ for some $g \in \mathcal{G}$, and there are $w, v \in \mathcal{B}$, such that

$$\mathbf{LM}(f) = \mathbf{LM}(\widetilde{f}) = w\mathbf{LM}(\widetilde{g})v = w\mathbf{LM}(g)v,$$

i.e., $\mathbf{LM}(g) \mid \mathbf{LM}(f)$. This shows that $\mathcal{G}$ is a Gröbner basis for I in R. $\square$

We call the Gröbner basis $\widetilde{\mathcal{G}}$ obtained in Theorem 3.3, the *noncentral homogenization* of $\mathcal{G}$ in $K\langle X, T\rangle$ with respect to T, or $\widetilde{\mathcal{G}}$ is a *noncentral homogenized Gröbner basis* with respect to T.

By Theorem 1.3, Lemma 3.1, and Theorem 3.3, the following corollary is straightforward.

Corollary 3.4. *Let I be an arbitrary ideal of* $K\langle X\rangle$. *With notation as before, if* $\mathcal{G}$ *is a Gröbner basis of I with respect to the data* $(\mathcal{B}, \prec_{gr})$, *then, with respect to the data* $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, *we have*

$$N(\langle \widetilde{I}\rangle) = N(\widetilde{\mathcal{G}}) = \{T^r w \mid w \in N(\mathcal{G}),\ r \in \mathbb{N}\},$$

that is, the set $N(\langle \widetilde{I}\rangle)$ *of normal monomials in* $\widetilde{\mathcal{B}}(\operatorname{mod}\langle \widetilde{I}\rangle)$ *is determined by the set* $N(I) = N(\mathcal{G})$ *of normal monomials in* $\mathcal{B}$ (mod *I*). *Hence, the algebra* $K\langle X, T\rangle / \langle \widetilde{I}\rangle = K\langle X, T\rangle / \langle \widetilde{\mathcal{G}}\rangle$ *has the K-basis*

$$\overline{N(\langle \widetilde{I}\rangle)} = \{\overline{T^r w} \mid w \in N(I),\ r \in \mathbb{N}\}. \qquad \square$$

As with the central (de)homogenization with respect to the commuting variable *t* in Section 2, we may also obtain a Gröbner basis for an ideal *I* of $K\langle X\rangle$ by dehomogenizing a homogeneous Gröbner basis of the ideal $\langle \widetilde{I}\rangle \subset K\langle X, T\rangle$. Below, we give a more general approach to this assertion that generalizes essentially ([13], Theorem 5), in which $\mathscr{G}$ is taken to be a *reduced Gröbner basis*, and its proof depends on the reducibility of $\mathscr{G}$.

Theorem 3.5. *Let J be a graded ideal of* $K\langle X, T\rangle$, *and suppose that* $\{X_iT - TX_i \mid 1 \le i \le n\} \subset J$. *If* $\mathscr{G}$ *is a homogeneous Gröbner basis of J with respect to the data* $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, then $\mathscr{G}_\sim = \{G_\sim \mid G \in \mathscr{G}\}$ *is a Gröbner basis for the ideal* $J_\sim$ *in* $K\langle X\rangle$ *with respect to the data* $(\mathcal{B}, \prec_{gr})$.

Proof. If $\mathscr{G}$ is a Gröbner basis of *J*, then $\mathscr{G}$ generates *J* and hence, $\mathscr{G}_\sim = \psi(\mathscr{G})$ generates $J_\sim = \psi(J)$. We show next that for each nonzero $h \in J_\sim$, there is some $G_\sim \in \mathscr{G}_\sim$ such that $\mathbf{LM}(G_\sim) \mid \mathbf{LM}(h)$, and hence $\mathscr{G}_\sim$ is a Gröbner basis for $J_\sim$.

Since $\{X_iT - TX_i \mid 1 \le i \le n\} \subset J$, by Lemma 3.1(v), there exists a homogeneous element $H \in J$ and some $r \in \mathbb{N}$ such that $T^r (H_\sim)^\sim = H$ and $H_\sim = h$. It follows that

$$\mathbf{LM}(H) = T^{r}\mathbf{LM}((H_{\sim})^{\sim}) = T^{r}\mathbf{LM}(\widetilde{h}) = T^{r}\mathbf{LM}(h). \tag{1}$$

On the other hand, there is some $G \in \mathscr{G}$ such that $\mathbf{LM}(G) \mid \mathbf{LM}(H)$, i.e., there are $W, V \in \widetilde{\mathcal{B}}$ such that

$$\mathbf{LM}(H) = W\mathbf{LM}(G)V. \tag{2}$$

But, by the above (1), we must have $\mathbf{LM}(G) = T^{q}w$, for some $q \in \mathbb{N}$ and $w \in \mathcal{B}$. Thus, by Lemma 3.2(ii),

$$\mathbf{LM}(G_{\sim}) = w = \mathbf{LM}(G)_{\sim} \text{ w.r.t. } \prec_{gr} \text{ on } \mathcal{B}. \tag{3}$$

Combining (1), (2), and (3) above, we then obtain

$$\begin{aligned}\mathbf{LM}(h) &= \mathbf{LM}(H)_{\sim}\\ &= (W\mathbf{LM}(G)V)_{\sim}\\ &= W_{\sim}\mathbf{LM}(G)_{\sim}V_{\sim}\\ &= W_{\sim}\mathbf{LM}(G_{\sim})V_{\sim}.\end{aligned}$$

This shows that $\mathbf{LM}(G_{\sim}) \mid \mathbf{LM}(h)$, as expected. □

We call the Gröbner basis $\mathscr{G}_{\sim}$ obtained in Theorem 3.5, the *noncentral dehomogenization* of $\mathscr{G}$ in $K\langle X\rangle$ with respect to T, or $\mathscr{G}_{\sim}$ is a *noncentral dehomogenized Gröbner basis* with respect to T.

Corollary 3.6. *Let I be an ideal of $K\langle X\rangle$. If $\mathscr{G}$ is a homogeneous Gröbner basis of $\langle\widetilde{I}\rangle$ in $K\langle X, T\rangle$ with respect to the data $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, then $\mathscr{G}_{\sim} = \{g_{\sim} \mid g \in \mathscr{G}\}$ is a Gröbner basis for I in $K\langle X\rangle$ with respect to the data $(\mathcal{B}, \prec_{gr})$. Moreover, if I is generated by the subset F and $\widetilde{F} \subset \mathscr{G}$, then $F \subset \mathscr{G}_{\sim}$.*

Proof. Put $J = \langle\widetilde{I}\rangle$. Then since $J_{\sim} = I$, it follows from Theorem 3.5 that, if $\mathscr{G}$ is a homogeneous Gröbner basis of J, then $\mathscr{G}_{\sim}$ is a Gröbner basis for I. The second assertion of the theorem is clear by Lemma 3.1(ii).

□

Let S be a nonempty subset of $K\langle X\rangle$ and $I = \langle S\rangle$, the ideal generated by S. Then, with $\widetilde{S} = \{\widetilde{f} \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \le i \le n\}$, in general, $\langle \widetilde{S}\rangle \subsetneq \langle \widetilde{I}\,\rangle$ in $K\langle X, T\rangle$ (for instance, consider $S = \{Y^3 - XY - X - Y, Y^2 - X + 3\}$ in the free algebra $K\langle X, Y\rangle$ and $\widetilde{S}$ in $K\langle X, Y, T\rangle$ with respect to T). Again, as we did in the case dealing with (de)homogenized Gröbner bases with respect to the commuting variable t, we take this place to set up the procedure of getting a Gröbner basis for I, and hence, a Gröbner basis for $\langle \widetilde{I}\,\rangle$ by producing a homogeneous Gröbner basis of the graded ideal $\langle \widetilde{S}\rangle$.

Proposition 3.7. *Let $I = \langle S\rangle$ be the ideal of $K\langle X\rangle$ as fixed above. Suppose the ground field K is computable. Then, a Gröbner basis for I and a homogeneous Gröbner basis for $\langle \widetilde{I}\,\rangle$ may be obtained by implementing the following procedure*:

Step 1. *Starting with the initial subset*

$$\widetilde{S} = \{\widetilde{f} \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \le i \le n\},$$

compute a homogeneous Gröbner basis $\mathscr{G}$ for the graded ideal $\langle \widetilde{S}\rangle$ of $K\langle X, T\rangle$.

Step 2. *Noticing $\langle \widetilde{S}\rangle_{\sim} = I$, use Theorem 3.5 and dehomogenize $\mathscr{G}$ with respect to T in order to obtain the Gröbner basis $\mathscr{G}_{\sim}$ for I.*

Step 3. *Use Theorem 3.3 and homogenize $\mathscr{G}_{\sim}$ with respect to T in order to obtain the homogeneous Gröbner basis $(\mathscr{G}_{\sim})^{\sim}$ for the graded ideal $\langle \widetilde{I}\rangle$.* □

Based on Theorems 3.3 and 3.5, we are able to determine those homogeneous Gröbner bases in $K\langle X, T\rangle$ that correspond bijectively to all Gröbner bases in $K\langle X\rangle$.

A homogeneous element $F \in K\langle X, T\rangle$ is called *dh-closed*, if $(F_{\sim})^{\sim} = F$; a subset S of $K\langle X, T\rangle$ consisting of dh-closed homogeneous elements is called a *dh-closed homogeneous set*.

To better understand the dh-closed property introduced above for homogeneous elements in $K\langle X, T\rangle$, we characterize a dh-closed homogeneous element as follows.

Lemma 3.8. *With notation as before, for a nonzero homogeneous element $F \in K\langle X, T\rangle$, the following two properties are equivalent with respect to $(\mathcal{B}, \prec_{gr})$ and $(\widetilde{\mathcal{B}}, \prec_{T-gr})$:*

(i) *F is dh-closed;*

(ii) *$F = \sum_i \lambda_i T^{r_i} w_i$ satisfying $\mathbf{LM}(F_{\sim}) = \mathbf{LM}(F)$, where $\lambda_i \in K^*$, $r_i \in \mathbb{N}$, and $w_i \in \mathcal{B}$.*

Proof. This follows easily from Lemma 3.1(iii). □

Let $I = \langle S\rangle$ be the ideal of $K\langle X\rangle$ generated by a subset S. Recall that, we have defined

$$\widetilde{S} = \{\widetilde{f} \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \le i \le n\}.$$

If $\mathscr{H}$ is a nonempty dh-closed homogeneous set in $K\langle X, T\rangle$ such that the subset

$$\mathscr{G} = \mathscr{H} \cup \{X_iT - TX_i \mid 1 \le i \le n\},$$

forms a Gröbner basis for the graded ideal $J = \langle \mathscr{G}\rangle$ with respect to $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, then we call $\mathscr{G}$ a *dh-closed homogeneous Gröbner basis*.

Theorem 3.9. *With respect to the systems $(\mathcal{B}, \prec_{gr})$ and $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, there is an one-to-one correspondence between the set of all Gröbner bases in $K\langle X\rangle$, and the set of all dh-closed homogeneous Gröbner bases in $K\langle X, T\rangle$:*

$$\begin{array}{ccc} \{\textit{Gröbner bases } \mathcal{G} \textit{ in } K\langle X\rangle\} & \leftrightarrow & \left\{\begin{array}{c}\textit{dh-closed homogeneous}\\ \textit{Gröbner bases } \mathscr{G} \textit{ in } K\langle X, T\rangle\end{array}\right\}, \\ \mathcal{G} & \to & \widetilde{\mathcal{G}}, \\ \mathscr{G}_{\sim} & \leftarrow & \mathscr{G}, \end{array}$$

and this correspondence also gives rise to a bijective map between the set of all minimal Gröbner bases in $K\langle X\rangle$, *and the set of all dh-closed minimal homogeneous Gröbner bases in* $K\langle X, T\rangle$.

Proof. By the definitions of homogenization and dehomogenization with respect to T, Theorems 3.3 and 3.5, it can be verified directly that the given rule of correspondence defines an one-to-one map. By the definition of a minimal Gröbner basis, the second assertion follows from Lemmas 3.2(i) and 3.8(ii). □

Below, we characterize the graded ideal generated by a dh-closed homogeneous Gröbner basis in $K\langle X, T\rangle$. To make the argument more convenient, we let $\mathscr{C}$ denote the ideal of $K\langle X, T\rangle$ generated by the Gröbner basis $\{X_iT - TX_i \mid 1 \le i \le n\}$, and let K-span $N(\mathscr{C})$ denote the K-space spanned by the set $N(\mathscr{C})$ of normal monomials in $\widetilde{\mathcal{B}}\,(\mathrm{mod}\,\mathscr{C})$. Noticing that, with respect to the monomial ordering $\prec_{T-gr}$ on $\widetilde{\mathcal{B}}$, $\mathbf{LM}(X_iT - TX_i) = X_iT$ for all $1 \le i \le n$, so each element $F \in K$-span $N(\mathscr{C})$ is of the form $F = \sum_i \lambda_i T^{r_i} w_i$ with $\lambda_i \in K^*$, $r_i \in \mathbb{N}$, and $w_i \in \mathcal{B}$.

Theorem 3.10. *With the convention made above, let* $\mathscr{H} \subset K$-*span* $N(\mathscr{C})$ *be a subset consisting of nonzero homogeneous elements. Suppose that the subset* $\mathscr{G} = \mathscr{H} \cup \{X_iT - TX_i \mid 1 \le i \le n\}$ *forms a minimal Gröbner basis for the graded ideal* $J = \langle \mathscr{G}\rangle$ *in* $K\langle X, T\rangle$ *with respect to the data* $(\widetilde{\mathcal{B}}, \prec_{T-gr})$. *Then, the following statements are equivalent*:

(i) *$\mathscr{G}$ is a dh-closed homogeneous Gröbner basis, i.e., $\mathscr{H}$ is a dh-closed homogeneous set*;

(ii) *J has the property* $\langle (J_{\sim})^{\sim} \rangle = J$;

(iii) *$K\langle X, T\rangle / J$ is a T-torsionfree (left) $K\langle X, T\rangle$-module, i.e., if $\bar{f} = f + J \in K\langle X, T\rangle / J$ and $\bar{f} \neq 0$, then $T\bar{f} \neq 0$, or equivalently, $Tf \notin J$*;

(iv) $TK\langle X, T\rangle \cap J = TJ$.

Proof. (i) $\Rightarrow$ (ii) If $\mathscr{G}$ is dh-closed, then by Theorem 3.5, $\mathscr{G}_{\sim}$ is a Gröbner basis for the ideal $J_{\sim}$ with respect to $(\mathcal{B}, \prec_{gr})$. Furthermore, it follows from Theorem 3.3 that $\mathscr{G}$ is a Gröbner basis for $\langle (J_{\sim})^{\sim} \rangle$ with respect to $(\widetilde{\mathcal{B}}, \prec_{T-gr})$. Hence, $J = \langle \mathscr{G} \rangle = \langle (J_{\sim})^{\sim} \rangle$.

(ii) $\Rightarrow$ (iii) Noticing that J is a graded ideal and T is a homogeneous element in $K\langle X, T\rangle$, it is sufficient to show that T does not annihilate any nonzero homogeneous element of $K\langle X, T\rangle / J$. Suppose $F \in K\langle X, T\rangle_p$ and $TF \in J$. Then since $J = \langle (J_{\sim})^{\sim} \rangle$, we have

$$(F_{\sim})^{\sim} = ((TF)_{\sim})^{\sim} \in \langle (J_{\sim})^{\sim} \rangle = J. \tag{1}$$

Moreover, by Lemma 3.1(iii), there exist $L \in J$ and $r \in \mathbb{N}$ such that

$$F = L + T^{r}(F_{\sim})^{\sim}. \tag{2}$$

Hence, (1) + (2) yields $F \in J$, as desired.

(iii) $\Leftrightarrow$ (iv) Obvious.

(iii) $\Rightarrow$ (i) Let $H \in \mathscr{H} - \{X_i T - TX_i \mid 1 \le i \le n\}$. Then, $H = \sum_i \lambda_i T^{r_i} w_i \in K\text{-span}\, N(\mathscr{C})$ such that $H = T^{r}(H_{\sim})^{\sim}$ for some $r \in \mathbb{N}$. If $r \geq 1$, then since $K\langle X, T\rangle / J$ is T-torsionfree, we must have $(H_{\sim})^{\sim} \in J$, and $\mathbf{LM}(H) \nmid \mathbf{LM}((H_{\sim})^{\sim})$. Hence, there exists some $H' \in \mathscr{H} - \{H\}$ such that

$\mathbf{LM}(H') | \mathbf{LM}((H_\sim)^\sim)$ and consequently, $\mathbf{LM}(H') | \mathbf{LM}(H)$, contradicting the minimality of $\mathscr{G}$. Therefore, $r = 0$, i.e., $H = (H_\sim)^\sim$. This shows that $\mathscr{H}$ is dh-closed. □

Corollary 3.11. *With notation as before, let J be a graded ideal of $K\langle X, T\rangle$. If, with respect to $(\mathcal{B}, \prec_{gr})$ and $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, J has a dh-closed minimal homogeneous Gröbner basis, then every minimal homogeneous Gröbner basis of J is dh-closed.*

Proof. This follows from the fact that each of the properties (ii)-(iv) in Theorem 3.10 does not depend on the choice of the generating set for J. □

Let J be a graded ideal of $K\langle X, T\rangle$. If J has the property mentioned in Theorem 3.10(ii), i.e., $\langle (J_\sim)^\sim \rangle = J$, then we call J a *dh-closed graded ideal*. It is easy to see that, there is an one-to-one correspondence between the set of all ideals in $K\langle X\rangle$, and the set of all dh-closed graded ideals in $K\langle X, T\rangle$:

$$\begin{array}{ccc} \{\text{ideals } I \text{ in } K\langle X\rangle\} & \leftrightarrow & \{\text{dh-closed graded ideals } J \text{ in } K\langle X, T\rangle\}, \\ I & \rightarrow & \langle \widetilde{I} \rangle, \\ J_\sim & \leftarrow & J. \end{array}$$

Note that in principle, Gröbner bases are computable in $K\langle X, T\rangle$, if the ground field K is computable. By the foregoing argument, to know whether a given graded ideal J of $K\langle X, T\rangle$ is dh-closed, it is sufficient to check, if J contains a minimal homogeneous Gröbner basis of the form $\mathscr{G} = \mathscr{H} \cup \{X_iT - TX_i \mid 1 \le i \le n\}$, in which $\mathscr{H} \subset K\text{-span}\, N(\mathscr{C})$ is a dh-closed homogeneous set.

Finally, as in the end of Section 2, let us point out that, if we start with the free K-algebra $K\langle X_1, \dots, X_n\rangle$, then everything we have done in this section can be done with respect to each $X_i = T$, $1 \le i \le n$, that is, just work with $K\langle X\rangle = K\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\rangle$ and $K\langle X, T\rangle = K\langle X_1, \dots, X_n\rangle$ with $T = X_i$. Also, instead of mentioning a version of each result obtained before, we highlight the respective version of Theorems 3.5 and 3.9 in this case as follows.

Theorem 3.12. *For each* $X_i = T, 1 \le i \le n$, *let* $K\langle X\rangle = K\langle X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\rangle$, $K\langle X, T\rangle = K\langle X_1, \ldots, X_n\rangle$ *with* $T = X_i$, *and fix the admissible systems* $(\mathcal{B}, \prec_{gr}), (\widetilde{\mathcal{B}}, \prec_{T-gr})$ *for* $K\langle X\rangle$ *and* $K\langle X, T\rangle$, *respectively, as before. The following statements hold.*

(i) *If* J *is a graded ideal of* $K\langle X, T\rangle$ *that contains the subset* $\{X_jT - TX_j \mid j \ne i\}$, *and if* $\mathcal{G}$ *is a homogeneous Gröbner basis of* J *with respect to* $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, *then* $\mathcal{G}_{\sim} = \{g_{\sim} \mid g \in \mathcal{G}\}$ *is a Gröbner basis for the ideal* $J_{\sim}$ *in* $K\langle X\rangle$ *with respect to* $(\mathcal{B}, \prec_{gr})$.

(ii) *There is an one-to-one correspondence between the set of all dh-closed homogeneous Gröbner bases in* $K\langle X, T\rangle$ *and the set of all Gröbner bases in* $K\langle X\rangle$, *under which dh-closed minimal Gröbner bases correspond to minimal Gröbner bases.* □

4. Algebras Defined by dh-Closed Homogeneous Gröbner Bases

The characterization of dh-closed graded ideals in terms of dh-closed homogeneous Gröbner bases given in Section 2 and Section 3 indeed provides us with an effective way to study algebras defined by dh-homogeneous Gröbner bases, that is, such algebras can be studied as Rees algebras (defined by grading filtration) via studying algebras with simpler defining relations as demonstrated in ([6], [10], [11]). Below, we present details on this conclusion.

All notions and notations used in previous sections are maintained.

Let A be a K-algebra. Recall that an $\mathbb{N}$-filtration of A is a family $FA = \{F_pA\}_{p\in\mathbb{N}}$ with each F_pA a K-subspace of A, such that (1) $1 \in F_0A$; (2) $\bigcup_{p\in\mathbb{N}} F_pA = A$; (3) $F_pA \subseteq F_{p+1}A$ for all $p \in \mathbb{N}$; and $F_pAF_qA \subseteq F_{p+q}A$. If A has an $\mathbb{N}$-filtration FA, then FA determines two

$\mathbb{N}$-graded K-algebras $G(A) = \oplus_{p\in\mathbb{N}} G(A)_p$ with $G(A)_p = F_pA / F_{p-1}A$, and $\widetilde{A} = \oplus_{p\in\mathbb{N}} \widetilde{A}_p$ with $\widetilde{A}_p = F_pA$, where $G(A)$ is called the *associated graded algebra* of A and $\widetilde{A}$ is called the *Rees algebra* of A.

Let $R = \oplus_{p\in\mathbb{N}} R_p$ be an $\mathbb{N}$-graded K-algebra. Then R has the $\mathbb{N}$-grading filtration $FR = \{F_pR\}_{p\in\mathbb{N}}$ with $F_pR = \oplus_{i\le p} R_i$. If I is an ideal of R and $A = R / I$, then A has the $\mathbb{N}$-filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ induced by FR, i.e., $F_pA = (F_pR + I) / I$. For instance, if the commutative polynomial K-algebra $R = K[x_1, \dots, x_n]$ is equipped with the natural $\mathbb{N}$-gradation, i.e., each x_i has degree 1, or if the non-commutative free K-algebra $R = K\langle X_1, \dots, X_n\rangle$ is equipped with the natural $\mathbb{N}$-gradation, i.e., each X_i has degree 1, then the usually used natural $\mathbb{N}$-filtration FA on $A = R / I$ is just the filtration induced by the $\mathbb{N}$-grading filtration FR of R. Consider the polynomial ring $R[t]$ and the mixed $\mathbb{N}$-gradation of $R[t]$ as described in Section 2. By ([7], [9], [10]), or in a similar way as in loc. cit., it can be proved that there are graded K-algebra isomorphisms:

$$G(A) \cong R / \langle \mathbf{LH}(I) \rangle, \quad \widetilde{A} \cong R[t] / \langle I^* \rangle, \tag{1}$$

where $\mathbf{LH}(I) = \{\mathbf{LH}(f) \mid f \in I\}$ with $\mathbf{LH}(f)$ the $\mathbb{N}$-leading homogeneous element of f as defined in [10] (i.e., if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p - \{0\}$, $f_{p-i} \in R_{p-i}$, then $\mathbf{LH}(f) = f_p$), and $I^* = \{f^* \mid f \in I\}$ with f^*, the homogenization of f in $R[t]$ with respect to t. Now, suppose that R has a Gröbner basis theory with respect to some admissible system $(\mathcal{B}, \prec_{gr})$ as in Section 2, and let $J = \langle \mathscr{G} \rangle$ be a graded ideal of $R[t]$ generated by a dh-closed homogeneous Gröbner basis $\mathscr{G}$. Let I denote the dehomogenization ideal J_* of J in R with respect to t, i.e., $I = J_*$. Then by Theorems 2.5 and 2.3, we have

$$I = J_* = \langle \mathscr{G}_* \rangle, \quad \langle I^* \rangle = \langle (J_*)^* \rangle = \langle \mathscr{G} \rangle = J. \tag{2}$$

Furthermore, from ([7], [9], [10]), we know that $\mathrm{LH}(\mathscr{G}_*) = \{\mathrm{LH}(g_*) \mid g_* \in \mathscr{G}_*\}$ is a Gröbner basis for the graded ideal $\langle \mathrm{LH}(I)\rangle$ in R, and so

$$\langle \mathrm{LH}(I)\rangle = \langle \mathrm{LH}(\mathscr{G}_*)\rangle. \tag{3}$$

It follows from (1) + (2) + (3) that we have proved the following.

Proposition 4.1. *With notation and the assumption on R as above, putting $A = R/\langle \mathscr{G}_*\rangle$, then there are graded K-algebra isomorphisms*:

$$G(A) \cong R/\langle \mathrm{LH}(\mathscr{G}_*)\rangle, \quad \widetilde{A} \cong R[t]/\langle \mathscr{G}\rangle = R[t]/J. \qquad \square$$

Thus, the algebra $R[t]/\langle \mathscr{G}\rangle = \widetilde{A}$ can be studied via studying the algebras $R/\langle \mathscr{G}_*\rangle = A$ and $R/\langle \mathrm{LH}(\mathscr{G}_*)\rangle = G(A)$. For instance, $\widetilde{A}$ is semiprime (prime, a domain), if and only if A is semiprime (prime, a domain); if $G(A)$ is semiprime (prime, a domain), then so are A and $\widetilde{A}$; if $G(A)$ is Noetherian (artinian), then so are A and $\widetilde{A}$; if $G(A)$ is Noetherian with finite global dimension, then so are A and $\widetilde{A}$, etc. (see [6] for more details).

Turning to the free K-algebras $K\langle X\rangle = K\langle X_1, \ldots, X_n\rangle$ and the free K-algebra $K\langle X, T\rangle = \langle X_1, \ldots, X_n, T\rangle$, let the admissible system $(\mathcal{B}, \prec_{gr})$ for $K\langle X\rangle$, and the admissible system $(\widetilde{\mathcal{B}}, \prec_{T-gr})$ for $K\langle X, T\rangle$ be as fixed in Section 3.

Proposition 4.2. *With the convention made above, let $\mathscr{G}$ be a dh-closed homogeneous Gröbner basis in $K\langle X, T\rangle$ with respect to the data $(\widetilde{\mathcal{B}}, \prec_{T-gr})$, and put $A = K\langle X\rangle/\langle \mathscr{G}_{\sim}\rangle$. Considering the $\mathbb{N}$-filtration FA of A induced by the (weight) $\mathbb{N}$-grading filtration $FK\langle X\rangle$ of $K\langle X\rangle$, then there are graded K-algebra isomorphisms*:

$$G(A) \cong K\langle X\rangle/\langle \mathrm{LH}(\mathscr{G}_{\sim})\rangle, \quad \widetilde{A} \cong K\langle X, T\rangle/\langle \mathscr{G}\rangle,$$

where $\mathrm{LH}(\mathscr{G}_\sim) = \{\mathrm{LH}(g_\sim) \mid g_\sim \in \mathscr{G}_\sim\}$ *with* $\mathrm{LH}(g_\sim)$, *the* $\mathbb{N}$-*leading homogeneous element of* $g_\sim$ *with respect to the* $\mathbb{N}$-*gradation of* $K\langle X\rangle$ (*see an explanation above*).

Proof. To be convenient, let us put $J = \langle \mathscr{G} \rangle$ and $I = \langle \mathscr{G}_\sim \rangle$. By ([7], [9], [10]), there are graded K-algebra isomorphisms:

$$G(A) \cong K\langle X\rangle / \langle \mathrm{LH}(I)\rangle, \quad \widetilde{A} \cong K\langle X, T\rangle / \langle \widetilde{I} \rangle. \tag{1}$$

By Theorems 3.5 and 3.3, we have

$$I = \langle \mathscr{G}_\sim \rangle = J_\sim, \quad \langle \widetilde{I} \rangle = \langle (J_\sim)^\sim \rangle = \langle \mathscr{G} \rangle = J. \tag{2}$$

Furthermore, from ([7], [9], [10]), we know that $\mathrm{LH}(\mathscr{G}_\sim)$ is a Gröbner basis for the graded ideal $\langle \mathrm{LH}(I)\rangle$ in $K\langle X\rangle$, and so

$$\langle \mathrm{LH}(I)\rangle = \langle \mathrm{LH}(\mathscr{G}_\sim)\rangle. \tag{3}$$

It follows from (1) + (2) + (3) that, the desired algebra isomorphisms are established. □

Thus, as demonstrated in ([10], [11]), the algebra $K\langle X, T\rangle / \langle \mathscr{G} \rangle = \widetilde{A}$ can be studied via studying the algebras $K\langle X\rangle / \langle \mathscr{G}_\sim \rangle = A$, $K\langle X\rangle / \langle \mathrm{LH}(\mathscr{G}_\sim)\rangle = G(A)$, and the monomial algebra $K\langle X\rangle / \langle \mathrm{LH}(\mathscr{G}_\sim)\rangle$. The reader is referred to loc. cit. for more details.

Remark. Note that the foregoing Theorems 2.12 and 3.12 have actually provided us with a practical stage to bring Propositions 4.1 and 4.2 into play.

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